Problem set 8: Optimal Routing

1 Routing and duality

Recall from lecture that a routing problem is specified by the list of CFMMs that the trader can interact with and a concave, increasing utility function $U : \mathbf{R}^n \to \mathbf{R}$, depending only on the *net output* of all trades. This utility function encodes what the trader wants to do (*e.g.*a swap or an arbitrage).

Each CFMM is associated with a subset of n_i tokens which we will write as some subset $S_i \subseteq \{1, \dots, n\}$. Every CFMM i has some trading function $\varphi_i : \mathbf{R}^{n_i} \to \mathbf{R} \cup \{\infty\}$, reserves $R_i \in \mathbf{R}^{n_i}$, fee $0 < \gamma_i \le 1$, and matrix $A_i \in \mathbf{R}^{n \times n_i}$, mapping the local token basket that the CFMM trades to the network's token basket. We solve for the tendered baskets $\Delta_i \in \mathbf{R}_+^{n_i}$, the received baskets $\Lambda_i \in \mathbf{R}_+^{n_i}$ for each CFMM i, which results in the network trade vector $\Psi \in \mathbf{R}^n$ that maximizes the function U. The optimal routing problem is

$$\begin{array}{ll} \text{maximize} & U(\Psi) \\ \text{subject to} & \Psi = \sum_{i=1}^m A_i(\Lambda_i - \Delta_i) \\ & \varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i) \ge \varphi_i(R_i), \quad i = 1, \dots, m \\ & \Delta_i \ge 0, \quad \Lambda_i \ge 0, \quad i = 1, \dots, m. \end{array} \tag{1}$$

In this question, we will explore some mathematical properties of the routing problem.

a) Show that the optimality conditions for (1) are

$$\begin{aligned} &\nabla U(\Psi) = \nu, \\ &\gamma_i \lambda_i \nabla \varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i) \le A_i^T \nu \le \lambda_i \nabla \varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i), \quad i = 1, \dots, m, \end{aligned}$$

Hint. Refer to [BV04, §5.5]. It may be helpful to rewrite (1) as

$$\begin{array}{ll} \text{minimize} & -U(\Psi) \\ \text{subject to} & \Psi = \sum_{i=1}^m A_i(\Lambda_i - \Delta_i) \\ & \varphi_i(R_i) - \varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i) \le 0, \quad i = 1, \dots, m \\ & -\Delta_i \le 0, \quad -\Lambda_i \le 0, \quad i = 1, \dots, m. \end{array}$$

b) Combine these conditions into one inequality and provide an interpretation in terms of marginal utilities.

Hint. Recall that $\nabla \varphi(R)$ are the unscaled prices of the assets in a CFMM with trading function φ at reserves R.

c) Consider a linear utility function $U(\Psi) = c^T \Psi$. What are the conditions under which we do not make a trade? Compare these to the ones in Problem Set 3 and to the ones above.

More generally, the optimality conditions for (1) are equivalent to those for the optimal arbitrage problem with a linear utility function and a particular choice of c: $c = \nabla U(\Psi)$.

This perspective is very appealing: if we know this c, we can optimally arbitrage each CFMM independently and in parallel. This insight suggests constructing an algorithm which iteratively updates the "true prices" c (equivalently, the true marginal utilities) instead of the trades Δ_i, Λ_i. In optimization parlance, this is known as a dual decomposition approach. The Julia package `CFMMRouter.jl`[1] implements an efficient algorithm for routing using this approach.

References

[BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[1] https://github.com/bcc-research/CFMMRouter.jl