Problem set 3: Constant function market makers

In this problem set, we'll show a few of the results hinted at in lecture, which we use throughout the remainder of the course.

1 Saturating equalities

In this problem, we show two short but important results for CFMMs.

Trading function. As a quick recap, a constant function market maker is defined by its *trading function*, $\varphi : \mathbf{R}^n_+ \to \mathbf{R}$, a *trading fee*, $0 < \gamma \le 1$, and its *reserves*, $R \in \mathbf{R}^n_+$. A proposed trade is specified by the user's *tendered basket*, $\Delta \in \mathbf{R}^n_+$ (which the user proposes to give to the CFMM) and the user's *received basket*, $\Lambda \in \mathbf{R}^n_+$ which the user proposes to receive from the CFMM. The entries of Δ and Λ denote the quantity of each token that the user proposes to tender or receive, respectively. The trade is *accepted* (or is acceptable) if

$$\varphi(R + \gamma\Delta - \Lambda) \ge \varphi(R).$$

If a trade is accepted, then Δ is taken from the user and Λ is paid out to the user, resulting in the CFMM's new reserves $R \leftarrow R + \Delta - \Lambda$. For the remainder of this problem, we will assume that φ is nondecreasing, and, for simplicity, we will assume that it is continuous. (As a side note, only upper semicontinuity is needed.)

Reasonable trades. We will say an acceptable trade, denoted $\Delta, \Lambda \in \mathbf{R}^n_+$, is *unreasonable* if there exists an acceptable trade $\Delta' \in \mathbf{R}^n_+$ and $\Lambda' \in \mathbf{R}^n_+$ satisfying

$$\Lambda' - \Delta' \ge \Lambda - \Delta,$$

with at least one entry not at equality. In other words, a trade is unreasonable if there is an acceptable trade that, after netting out, yields strictly more of at least one token. We will call a trade *reasonable* if it is not unreasonable.

Disjoint support. If $\gamma < 1$, show that $\Delta_i \Lambda_i = 0$, for $i = 1, \dots, n$, where Δ, Λ is any reasonable trade. (*Hint.* Show that, if both $\Delta_i > 0$ and $\Lambda_i > 0$, you can increase $\Lambda_i - \Delta_i$ while not changing $R_i + \gamma\Delta_i - \Lambda_i$.)

Constrants at equality. Show that, with any fee $0 < \gamma \leq 1$, any reasonable trade Δ, Λ will satisfy

$$\varphi(R + \gamma\Delta - \Lambda) = \varphi(R).$$

(*Hint.* Assume otherwise then use the continuity of the function φ to show such a trade cannot be reasonable.)

2 General no arbitrage condition

As a reminder, the *no-arbitrage* problem against an external market with prices $c \in \mathbf{R}^n_+$ is

$$\begin{array}{ll} \text{maximize} & c^T(\Lambda - \Delta) \\ \text{subject to} & \varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R) \\ & \Delta, \Lambda \geq 0, \end{array}$$

with variables $\Delta, \Lambda \in \mathbf{R}^n$. Assuming the trading function φ is differentiable and concave, use the optimality conditions at $\Delta = \Lambda = 0$ to show that the optimal value of this problem is zero if, and only if,

$$\gamma\nabla\varphi(R) \leq \alpha c \leq \nabla\varphi(R),$$

for some $\alpha \geq 0$. In other words, an arbitrageur will not perform arbitrage with the CFMM if the CFMM's marginal prices (defined as $\nabla\varphi(R)$ in lecture) are within some bound γ of the external market price c. (Here, the multiplier α comes in from the fact that only the relative prices between assets matter.)

(*Hint.* Let's say we are given a problem of the form:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & g(x) \geq 0 \\ & x \geq 0, \end{array}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$ are differentiable concave functions and $x \in \mathbf{R}^n$ is the optimization variable. The optimality conditions for this problem are that there exists some point $x^\star \in \mathbf{R}^n$ and nonnegative multiplier $\lambda \geq 0$ satisfying:

$$\nabla f(x^\star) + \lambda\nabla g(x^\star) \leq 0.$$

These conditions are both necessary and sufficient for $x^\star$ to be an optimal solution to the problem above. As a second hint, let $x = (\Delta, \Lambda)$ and apply these optimality conditions to the problem above. For more information on the optimality conditions, see Boyd and Vandenberghe's *Convex Optimization* Chapter 5, and, more specifically, §5.5.3.)